ON THE CONVERGENCE AND DIFFERENTIATION OF TRIGONOMETRIC SERIES

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In this paper are given extensions of two theorems, one due to Schlömilch,* relating to the convergence, the other due to Lerch,† relating to the differentiation of trigonometric series, the mode of proof being adapted to the case in hand from an article on "Fourier's Series" by Professor Bôcher.‡ Some examples are appended.

1. Consider the trigonometric series

$$\sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1}$$

A sufficient condition for the convergence of series (1) is the absolute convergence of the series of its coefficients. A more general sufficient condition is given by the following theorem:

The series (1) *converges uniformly in any interval which does not include or reach up to a root of the equation*

$$\cos \frac{h}{2}(x - t) = 0,$$

where h is a fixed integer and t any real constant, provided that

$$(\alpha) \qquad \lim a_n = 0, \qquad \lim b_n = 0$$

and the series

$$(\beta) \qquad \sum (|A_n| + |B_n|)$$

converges, where

$$A_n = (a_{n-h} - a_n)\sin \tfrac{1}{2} ht + (b_{n-h} + b_n)\cos \tfrac{1}{2} ht,$$
$$B_n = (a_{n-h} + a_n)\cos \tfrac{1}{2} ht - (b_{n-h} - b_n)\sin \tfrac{1}{2} ht.$$

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* *Compendium der höheren Analysis*, vol. 1, §40.

† *Annales de l'École normale supérieure*, ser. 3, vol. 12 (1895), p. 351.

‡ Annals of Mathematics, vol. 7, nos. 2 and 3, 1906.

Special cases of (β), when $t=\pi/h$ and $t=0$ respectively, are the convergence of

$$(\beta_1) \qquad \sum_1^\infty (|a_{n-h}-a_n|+|b_{n-h}-b_n|)$$

or

$$(\beta_2) \qquad \sum_1^\infty (|a_{n-h}+a_n|+|b_{n-h}+b_n|).$$

As a particular case under (β_1) may be noted the condition

$$(\beta_3) \qquad a_{n-h} \geqq a_n, \qquad b_{n-h} \geqq b_n; \qquad n \geqq N.$$

This reduces to the theorem of Schlömilch when $h=1$.

For the proof, let $S_k(x)$ denote the first k terms of (1) and form the following expression in which all a's and b's with negative or zero subscripts are to be put equal to zero:

$$(2) \quad 2\cos\frac{h}{2}(x-t)S_k(x)$$

$$=\sum_1^k\left[2a_n\cos nx\cos\frac{h}{2}(x-t)+2b_n\sin nx\cos\frac{h}{2}(x-t)\right]$$

$$=\sum_1^k\left\{a_n\left[\cos(nx+\frac{h}{2}x-\frac{h}{2}t)+\cos(nx-\frac{h}{2}x+\frac{h}{2}t)\right]\right.$$

$$\left.+b_n\left[\sin(nx+\frac{h}{2}x-\frac{h}{2}t)+\sin(nx-\frac{h}{2}x+\frac{h}{2}t)\right]\right\}$$

$$=\sum_{1+h}^{k+h}\left[a_{n-h}\cos(nx-\frac{h}{2}x-\frac{h}{2}t)+b_{n-h}\sin(nx-\frac{h}{2}x-\frac{h}{2}t)\right]$$

$$+\sum_1^k\left[a_n\cos(nx-\frac{h}{2}x+\frac{h}{2}t)+b_n\sin(nx-\frac{h}{2}x+\frac{h}{2}t)\right]$$

$$=\sum_1^k\left\{\left[(a_{n-h}-a_n)\sin\frac{h}{2}t+(b_{n-h}+b_n)\cos\frac{h}{2}t\right]\sin(n-\frac{h}{2})x\right.$$

$$\left.+\left[(a_{n-h}+a_n)\cos\frac{h}{2}t-(b_{n-h}-b_n)\sin\frac{h}{2}t\right]\cos(n-\frac{h}{2})x\right\}$$

$$+R_k(x),$$

where

$$R_k(x) = \sum_{k+1}^{k+h} \Big[(a_{n-h} \sin \frac{h}{2} t + b_{n-h} \cos \frac{h}{2} t) \sin (n - \frac{h}{2}) x$$
$$+ (a_{n-h} \cos \frac{h}{2} t - b_{n-h} \sin \frac{h}{2} t) \cos (n - \frac{h}{2}) x \Big]$$

By use of conditions (α), $\lim R_k(x) = 0$ uniformly, and by (β), the last Σ in (2) converges uniformly; the same is true after dividing by $2 \cos \frac{1}{2} h(x - t)$; hence $\lim S_k(x)$, or series (1), converges uniformly, in any interval which does not reach up to a root of the equation $\cos \frac{1}{2} h(x - t) = 0$.

2. Sufficient conditions for establishing the existence of the first derivative of series (1) are obtained in the following theorem, which also provides the means for writing this derivative.

Take h any integer, t a real constant, and let c_1 and c_2 be two points lying between two successive roots of the equation

$$\cos \frac{h}{2}(x - t) = 0.$$

(γ) *Suppose series* (1) *convergent at one point c, $c_1 \leqq c \leqq c_2$.*

(δ) *Assume the uniform convergence of*

$$\sum_1^\infty \Big\{ \Big[-\Big((n-h)a_{n-h} + na_n\Big) \cos \frac{h}{2} t + \Big((n-h)b_{n-h} - nb_n\Big) \sin \frac{h}{2} t \Big] \sin (n - \frac{h}{2}) x$$
$$+ \Big[\Big((n-h)a_{n-h} - na_n\Big) \sin \frac{h}{2} t + \Big((n-h)b_{n-h} + nb_n\Big) \cos \frac{h}{2} t \Big] \cos (n - \frac{h}{2}) x \Big\}$$

in the interval $c_1 \leqq x \leqq c_2$.

(ϵ) *Lastly, let* $\lim a_n = \lim b_n = 0$.

Then in this interval, series (1) *converges uniformly, represents a continuous function $f(x)$, and has a first derivative $f'(x)$ given by dividing series* (δ) *by $2 \cos \frac{1}{2} h(x - t)$.*

As before, all coefficients with negative or zero subscripts are to be put equal to zero.

In practice the following special forms, obtained from (δ) when $t = \frac{\pi}{h}$ and $t = 0$ respectively, are more useful for calculating the derivative:

$$(\delta_1) \qquad f'(x) = \frac{1}{2\sin\frac{1}{2}hx}\sum_1^\infty\Big\{\Big[(n-h)a_{n-h} - na_n\Big]\cos(n-\frac{h}{2})x + \Big[(n-h)b_{n-h} - nb_n\Big]\sin(n-\frac{h}{2})x\Big\},$$

$$(\delta_2) \qquad f'(x) = \frac{1}{2\cos\frac{1}{2}hx}\sum_1^\infty\Big\{-\Big[(n-h)a_{n-h} + na_n\Big]\sin(n-\frac{h}{2})x + \Big[(n-h)b_{n-h} + nb_n\Big]\cos(n-\frac{h}{2})x\Big\}.$$

To obtain this theorem, let $S_k'(x)$ denote the derivative of $S_k(x)$, then

$$(4) \qquad 2\cos\frac{h}{2}(x-t)S_k'(x)$$

$$= \sum_1^k\Big[-2na_n\sin nx\cos\frac{h}{2}(x-t) + 2nb_n\cos nx\cos\frac{h}{2}(x-t)\Big]$$

$$= \sum_1^k\Big\{-na_n\Big[\sin(nx+\frac{h}{2}x-\frac{h}{2}t) + \sin(nx-\frac{h}{2}x+\frac{h}{2}t)\Big] + nb_n\Big[\cos(nx+\frac{h}{2}x-\frac{h}{2}t) + \cos(nx-\frac{h}{2}x+\frac{h}{2}t)\Big]\Big\}$$

$$= \sum_{1+h}^{k+h}\Big[-(n-h)a_{n-h}\sin(nx-\frac{h}{2}x-\frac{h}{2}t) + (n-h)b_{n-h}\cos(nx-\frac{h}{2}x-\frac{h}{2}t)\Big] + \sum_1^k\Big[-na_n\sin(nx-\frac{h}{2}x+\frac{h}{2}t) + nb_n\cos(nx-\frac{h}{2}x+\frac{h}{2}t)\Big]$$

$$= \sum_1^k\Big\{\Big[-\Big((n-h)a_{n-h}+na_n\Big)\cos\frac{h}{2}t + \Big((n-h)b_{n-h}-nb_n\Big)\sin\frac{h}{2}t\Big]\sin(n-\frac{h}{2})x + \Big[\Big((n-h)a_{n-h}-na_n\Big)\sin\frac{h}{2}t + \Big((n-h)b_{n-h}+nb_n\Big)\cos\frac{h}{2}t\Big]\cos(n-\frac{h}{2})x\Big\} + R_k(x),$$

where

$$R_k(x) = \sum_{k+1}^{k+h} \left\{ \left[-(n-h) a_{n-h} \cos \frac{h}{2} t + (n-h) b_{n-h} \sin \frac{h}{2} t \right] \sin \left(n - \frac{h}{2}\right) x \right.$$

$$\left. + \left[(n-h) a_{n-h} \sin \frac{h}{2} t + (n-h) b_{n-h} \cos \frac{h}{2} t \right] \cos \left(n - \frac{h}{2}\right) x \right\}.$$

Let $G_k(x)$ denote the continuous function obtained by dividing each term of the last Σ in (4) by $2 \cos \frac{1}{2} h (x - t)$. Then from (4),

$$(5) \qquad S_k(x) = S_k(c) + \int_c^x G_k(x)\, dx + \int_c^x \frac{R_k(x)}{2 \cos \frac{1}{2} h (x-t)}\, dx.$$

We will now show that each of the three terms on the right of (5) approaches a limit uniformly for all values of x as k becomes infinite. This is true of the first term by (γ) ; call this limit C. By (δ) the second term approaches uniformly $\int_c^x G(x) dx$, if we denote by $G(x)$ the value of the series (δ) divided by $2 \cos \frac{1}{2} h (x - t)$. In the third term, integrate by parts, first replacing $R_k(x)$ by its value, and putting $(n - h)\, A_{n-h}$ and $(n - h)\, B_{n-h}$ in place of the expressions in the brackets. We obtain,

$$\int_c^x \frac{R_k(x)}{2 \cos \frac{1}{2} h (x-t)}\, dx$$

$$= \sum_{k+1}^{k+h} \int_c^x (n-h) \left[\frac{A_{n-h} \sin (n - \frac{1}{2} h) x + B_{n-h} \cos (n - \frac{1}{2} h) x}{2 \cos \frac{1}{2} h (x - t)} \right] dx$$

$$= \sum_{k+1}^{k+h} \frac{n-h}{n - \frac{1}{2} h} \left\{ \left[\frac{-A_{n-h} \cos (n - \frac{1}{2} h) x + B_{n-h} \sin (n - \frac{1}{2} h) x}{2 \cos \frac{1}{2} h (x-t)} \right]_c^x \right.$$

$$\left. - \frac{h}{2} \int_c^x \left[A_{n-h} \cos \left(n - \frac{h}{2}\right) x - B_{n-h} \sin \left(n - \frac{h}{2}\right) x \right] \frac{\sin \frac{1}{2} h (x-t)}{2 \cos^2 \frac{1}{2} h (x-t)}\, dx \right\}.$$

Take k so large that $\dfrac{n-h}{n - \frac{1}{2} h} < 2$, and take K so that $0 < K \leqq \cos^2 \dfrac{h}{2} (x - t)$; also note that $|x - c| < 2\pi$. Then

$$\left| \int_c^x \frac{R_k(x)}{2 \cos \frac{1}{2} h (x-t)}\, dx \right| \leqq \frac{2 + \frac{1}{2} h \pi}{K} \sum_{k+1}^{k+h} \Big[\, | A_{n-h} | + | B_{n-h} | \, \Big].$$

Then by (ϵ),

$$\lim_{k=\infty} \int_c^x \frac{R_k(x)}{2\cos\frac{1}{2}h(x-t)}\, dx = 0, \text{ uniformly.}$$

Having thus shown that the three terms on the right hand side of (5) approach limits uniformly, it follows that $S_k(x)$ approaches a limit, which we will call $f(x)$, uniformly, and

$$f(x) = C + \int_c^x G(x)\,dx,$$

hence,

$$f'(x) = G(x).$$

This is the desired derivative, and gives the equations (δ_1) and (δ_2) for the values of t indicated above.

The usual rule for differentiating the series (1), requiring the uniform convergence of the derived series, is the special case $h = 0$. Lerch's theorem is contained in (δ_1), when $h = 2$.

3. In conclusion, we consider two numerical examples.

Example 1. $$\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots.$$

Conditions (α) and (β_3) are satisfied, for every value of h. The largest interval of convergence is given by the value $h = 1$, whence the series converges uniformly between the roots of $\sin\frac{1}{2}x = 0$.

The derivative is given by (δ_1) where $h = 1$, and is $f'(x) = -\frac{1}{2}$. Then the sum of the series is

$$f(x) = \frac{2k+1}{2}\pi - \frac{x}{2}, \quad 2k\pi < x < (2k+2)\pi, \quad k = 0, 1, 2, \cdots,$$

the constant being obtained by putting $x = (2k+1)\pi$ in the given series.

Example 2. $$\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots.$$

When $h = 2$, (β_2) becomes

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots < 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Hence the series converges uniformly between the roots of $\cos x = 0$.

By formula (δ_2), when $h = 2$,

$$f'(x) = \frac{1 + \cos x}{2 \cos x},$$

whence

$$f(x) = \frac{1}{2}\left[x + \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - k\pi - \log\tan\frac{2k+1}{4}\pi\right]$$

in the intervals

$$(2k-1)\frac{\pi}{2} < x < (2k+1)\frac{\pi}{2}, \quad k = 0, \pm 1, \pm 2, \cdots.$$

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